

A Dirichlet unit theorem for Drinfeld modules

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Abstract We show that the module of integral points on a Drinfeld module satisfies an analogue of Dirichlet’s unit theorem, despite its failure to be finitely generated. As a consequence, we obtain a construction of a canonical finitely generated sub-module of the module of integral points. We use the results to give a precise formulation of a conjectural analogue of the class number formula.

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1 Introduction

Let F be a number field. Consider the exponential exact sequence

$$(2\pi i\mathbf{Z})^{r_2} \hookrightarrow (F \otimes_{\mathbf{Q}} \mathbf{R})^{\mathrm{Tr}=0} \xrightarrow{\exp} (F \otimes_{\mathbf{Q}} \mathbf{R})^{\times, \mathrm{Nm}=\pm 1} \twoheadrightarrow \{\pm 1\}^{r_1}, \quad (1)$$

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where

$$\exp(z) = \sum_{i \geq 0} \frac{z^i}{i!}$$

is the usual exponential function and r_1 (resp. r_2) is the number of real (resp. complex) places of F . Denote the ring of integers of F by O_F . The sequence (1) induces an exact sequence

$$(2\pi i \mathbf{Z})^{r_2} \hookrightarrow \exp^{-1}(O_F^\times) \rightarrow O_F^\times \rightarrow \{\pm 1\}^{r_1},$$

and Dirichlet's unit theorem is equivalent to the statement that

$$\exp^{-1}(O_F^\times) \subset (F \otimes_{\mathbf{Q}} \mathbf{R})^{\text{Tr}=0}$$

is discrete and co-compact (see e.g. [11, I.§8]).

Poonen [8] has shown that the module of integral points on a Drinfeld module of positive rank is not finitely generated. Nevertheless, we will show that it satisfies an analogue of the above formulation of Dirichlet's unit theorem. The statement will be given in the next section, the proof is in Sect. 3.

In Sect. 4 we discuss the results in the special case of the Carlitz module, which is in many ways the proper function field analogue of the multiplicative group, and in Sect. 5 we state a conjectural analogue of the class number formula and some evidence for it.

2 Statement

Let k be a finite field of q elements and let K be a finite extension of the rational function field $k(t)$. The integral closure of $k[t]$ in K will always be denoted by R .

Denote the q -th power Frobenius endomorphism of the additive group $\mathbf{G}_{a,R}$ by τ . We denote by E the additive group $\mathbf{G}_{a,R}$ equipped with an action φ of $k[t]$ given by a k -algebra homomorphism

$$\varphi: k[t] \rightarrow \text{End}(\mathbf{G}_{a,R}): t \mapsto t + a_1 \tau + \cdots + a_n \tau^n,$$

where $a_i \in R$ and $a_n \neq 0$. So E is a model over R of a Drinfeld module of rank n over K . For any R -algebra B we denote by $E(B)$ the module of B -rational points on E . This is nothing but the additive group of B equipped with the $k[t]$ -module structure given by φ . We do not exclude the case $n = 0$.

We denote the tangent space at zero of E by Lie_E . For any R -algebra B we have that $\text{Lie}_E(B) = B$ on which the action of t induced by $\varphi(t)$ is just multiplication by t .

Put $K_\infty := K \otimes_{k(t)} k((t^{-1}))$, let $k((t^{-1}))^{\text{sep}}$ be a separable closure of $k((t^{-1}))$ and put $K_\infty^{\text{sep}} := K \otimes_{k(t)} k((t^{-1}))^{\text{sep}}$. Note that K_∞ and K_∞^{sep} are products of fields.

There exists a unique power series

$$\exp_E(X) = \sum_{i=0}^{\infty} e_i X^{q^i} \in K[[X]]$$

with $e_0 = 1$ and such that

$$\varphi(t) \exp_E(X) = \exp_E(tX).$$

Drinfeld [3, §3] has shown that this power series converges on all of K_∞^{sep} and that it fits in a short exact sequence of $k[t]$ -modules

$$\Lambda_E \hookrightarrow \text{Lie}_E(K_\infty^{\text{sep}}) \xrightarrow{\exp_E} E(K_\infty^{\text{sep}})$$

where the $k[t]$ -module Λ_E is discrete in $\text{Lie}_E(K_\infty^{\text{sep}})$, and free of rank n times the separable degree of K over $k(t)$.

This sequence is $G = \text{Gal}(k((t^{-1}))^{\text{sep}}/k((t^{-1})))$ -equivariant, and taking invariants gives an exact sequence

$$\Lambda_E^G \hookrightarrow \text{Lie}_E(K_\infty) \xrightarrow{\exp_E} E(K_\infty) \rightarrow H^1(G, \Lambda_E), \quad (2)$$

where the surjectivity of the connecting map $E(K_\infty) \rightarrow H^1(G, \Lambda_E)$ follows from “additive Hilbert 90” [9, x.§1, prop. 1]. In contrast with the case of the multiplicative group (1), the group $H^1(G, \Lambda_E)$ is not finitely generated if $n > 0$.

Theorem 1 *The cokernel of $E(R) \rightarrow H^1(G, \Lambda_E)$ is finite. The inverse image under \exp of $E(R)$ is a discrete and co-compact sub- $k[t]$ -module of $\text{Lie}_E(K_\infty)$.*

Corollary 1 *The kernel of $E(R) \rightarrow H^1(G, \Lambda_E)$ is finitely generated.*

This suggests that the proper analogy is not so much between $E(R)$ and O_F^\times , but rather between the complex

$$E(R) \longrightarrow H^1(G, \Lambda_E)$$

and the complex

$$O_F^\times \longrightarrow \{\pm 1\}^{r_1}.$$

It is quite surprising that Theorem 1 actually holds in case E is of rank two or more. I do not know anything in the analogy between Drinfeld modules and elliptic curves that hints at such a result.

3 Proof

Lemma 1 *There exist compact open subgroups $L \subset \text{Lie}(K_\infty)$ and $L' \subset E(K_\infty)$ such that \exp_E maps L isomorphically onto L' .*

Proof Identify $\text{Lie}_E(K_\infty)$ and $E(K_\infty)$ with K_∞ in the obvious way. Define $\|\cdot\| : K_\infty \rightarrow \mathbf{R}$ as the maximum of the normalized absolute values on the components of K_∞ . Since

$$\exp_E(x) = x + \sum_{i=1}^{\infty} e_i x^{q^i}$$

is an entire function we have that $\|e_i\|$ tends to zero. We can therefore apply the non-archimedean implicit function theorem (see for example [5, 2.2]) to deduce the lemma.

Remark 1 Note that under the identification $\mathrm{Lie}_E(K_\infty) = K_\infty = E(K_\infty)$ we have that $L = L'$.

Proof of Theorem 1 We first prove that $\exp_E^{-1}(E(R))$ is discrete in $\mathrm{Lie}_E(K_\infty)$.

Assume that $\lambda_1, \lambda_2, \dots$ is a sequence of elements of $\exp_E^{-1}(E(R))$ that converges to 0. Then $(\exp_E(\lambda_i))_i$ converges to 0, but since $E(R)$ is discrete in $E(K_\infty)$, it follows that $\exp_E(\lambda_i) = 0$ for all i sufficiently large. Thus $\lambda_i \in \Lambda_E$ for all i sufficiently large, and as Λ_E is discrete in $\mathrm{Lie}_E(K_\infty)$ we have that $\lambda_i = 0$ for all i sufficiently large, and we conclude that $\exp_E^{-1}(E(R))$ is discrete in $\mathrm{Lie}_E(K_\infty)$.

Now we show that the cokernel of $E(R) \rightarrow H^1(G, \Lambda_E)$ is finite. Let L and L' be as in Lemma 1. There exists a finite k -vector space $V' \subset E(K_\infty)$ such that $L', E(R)$ and V' together span $E(K_\infty)$ as a k -vector space. It follows that $H^1(G, \Lambda_E)$ is spanned by the images of $E(R)$ and V' , and in particular that the cokernel of $E(R) \rightarrow H^1(G, \Lambda_E)$ is finite.

Finally the co-compactness. Let $V \subset \mathrm{Lie}_E(K_\infty)$ be a finite sub- k -vector space so that $\exp_E(V)$ contains $V' \cap \exp_E(\mathrm{Lie}_E(K_\infty))$. Then the subspaces L, V and $\exp_E^{-1}(E(R))$ span $\mathrm{Lie}_E(K_\infty)$, and since L and V are compact it follows that $\exp_E^{-1}(E(R))$ is co-compact in $\mathrm{Lie}_E(K_\infty)$.

4 The case of the Carlitz module

In this section E will always denote the Carlitz module, so $E = \mathbf{G}_{a,k[t]}$ with $k[t]$ -action given by

$$\varphi: t \mapsto t + \tau.$$

This Drinfeld module plays a role in function field arithmetic that is very similar to the role of the multiplicative group in number field arithmetic.

The exponential exact sequence becomes

$$k[t]\tilde{\pi} \hookrightarrow \mathrm{Lie}_E(k((t^{-1}))^{\mathrm{sep}}) \twoheadrightarrow E(k((t^{-1}))^{\mathrm{sep}})$$

with

$$\tilde{\pi} = \alpha^q \prod_{i=1}^{\infty} (1 - t^{1-q^i})^{-1},$$

where α is a chosen $(q-1)$ -st root of $-t$ (see [2, pp. 148–150], but note that Carlitz uses different normalisations). Of course the sub-module $k[t]\tilde{\pi}$ of $\mathrm{Lie}_E(k((t^{-1}))^{\mathrm{sep}})$ does not depend on the choice of α .

Denote the kernel of $E(R) \rightarrow H^1(G, \Lambda_E)$ by U_R .

Proposition 1 *Let E be the Carlitz module and R and K as above. Then U_R has rank $d - r$ where d is the separable degree of K over $k(t)$ and r is the number of places v of K above ∞ such that $-t$ has a $(q-1)$ -st root in K_v .*

Proof This follows immediately from Theorem 1 and the explicit description of Λ_E above.

For example, if $R = k[t]$ then U_R has rank 1 if $q > 2$ and rank 0 if $q = 2$.

Proposition 2 *Let E be the Carlitz module and R and K as above. Then all torsion elements of $E(R)$ are in U_R .*

Proof In fact, $H^1(G, \Lambda_E)$ is torsion-free, since there is a $k[G]$ -module V such that $\Lambda_E = V \otimes_k k[t]$, hence $H^1(G, \Lambda_E) = H^1(G, V) \otimes_k k[t]$.

Remark 2 In case K is a “cyclotomic” extension of $k(t)$, Anderson [1, pp. 194–196] has defined a finitely generated sub-module $\mathcal{L} \subset E(R)$ of “circular units”. Since these circular units are constructed as exponentials of elements in $\text{Lie}_E(K_\infty)$, one has $\mathcal{L} \subset U_R$, and comparing ranks one finds that the quotient is finite. It follows from Proposition 2 that U_R is in fact the divisible closure of \mathcal{L} in $E(R)$.

5 A conjectural class number formula

Finally we discuss a conjectural analogue of the class number formula. We continue with the notation of the previous section, in particular E denotes the Carlitz module over $k[t]$.

Definition 1 For a finite $k[t]$ -module M , we denote by $|M| \in k[t]$ the monic generator of the first fitting ideal of M .

This is a $k[t]$ -analogue of the cardinality of a finite abelian group. Explicitely, if $f_i \in k[t]$ are monic polynomials such that $M \cong \bigoplus_i k[t]/(f_i)$ then $|M| = \prod_i f_i$.

Definition 2 We define $\zeta_R(1) \in k((t^{-1}))$ by

$$\zeta_R(1) = \sum_{I \subset R} \frac{1}{|R/I|},$$

where I ranges over the non-zero ideals of R .

Note that in contrast with the classical (archimedean) harmonic series, this infinite sum converges.

Remark 3 This ad hoc definition of $\zeta_R(1)$ suffices for our purposes. Goss has defined an analogue $\zeta_R(s)$ of the Dedekind zeta function, of which $\zeta_R(1)$ is in fact a value. We refer to [4] for the details.

By Theorem 1 the natural map

$$\rho: \exp^{-1}(E(R)) \otimes_{k[t]} k((t^{-1})) \longrightarrow \text{Lie}_E(R) \otimes_{k[t]} k((t^{-1}))$$

induced by the inclusion $\exp_E^{-1}(E(R)) \rightarrow \text{Lie}_E(K_\infty)$ is an isomorphism of $k((t^{-1}))$ -vector spaces. Since both source and target have a natural $k[t]$ -module structure, the map has a well-defined determinant

$$\det(\rho) \in k((t^{-1}))^\times / k[t]^\times,$$

obtained by taking the determinant with respect to any chosen $k[t]$ -bases for $\exp^{-1}(E(R))$ and $\text{Lie}_E(R)$.

Definition 3 The *regulator* of R , denoted Reg_R is the unique monic representative in $k((t^{-1}))$ of $\det(\rho)$.

Remark 4 The invariant Reg_R of R is not quite the proper analogue of the classical regulator. The way we have defined it, it absorbs a part coming from $\ker(\exp)$ and also the square root of the discriminant. One should rather compare Reg_R with the product $R_F 2^{r_1} (2\pi)^{r_2} / w_F \sqrt{|\Delta_F|}$ for a number field F . However, since Theorem 1 lacks the “trace zero” part that occurs in the formulation of the Dirichlet unit theorem, it is impossible to make a precise analogy.

Definition 4 We denote the cokernel of $E(R) \rightarrow H^1(G, \Lambda_E)$ by H_R .

We can now state the conjectural analogue of the class number formula.

Conjecture 1 $\zeta_R(1) = \text{Reg}_R \cdot |H_R|$.

The remainder of this paper contains some evidence towards this conjecture, and some remarks on its interpretation.

Proposition 3 *Conjecture 1 is true if $R = k[t]$.*

Proof For $R = k[t]$ one verifies easily that $\exp^{-1}(E(R))$ is generated by $\log(1)$, where \log denotes the Carlitz logarithm, and that $E(R) \rightarrow H^1(G, \Lambda_E)$ is surjective. The conjecture then boils down to the identity $\zeta_R(1) = \log(1)$. This identity is already implicitly present in the work of Carlitz, see [2, p. 160], see also [1, p. 199].

The valuation of $\zeta_R(1)$ is zero, so also the right-hand side in the conjecture should have valuation zero. This is indeed the case:

Theorem 2 $v(\text{Reg}_R \cdot |H_R|) = 0$.

Before proving the theorem we give an algebraic description of the valuation of a “regulator”.

Lemma 2 *Let V be a finite dimensional $k((t^{-1}))$ -vector space, $L \subset V$ an open compact sub- k -space and $\Lambda_1, \Lambda_2 \subset V$ discrete and co-compact sub- $k[t]$ -modules. Let*

$$\rho: \Lambda_1 \otimes_{k[t]} k((t^{-1})) \xrightarrow{\sim} \Lambda_2 \otimes_{k[t]} k((t^{-1}))$$

be the induced isomorphism. For $i = 1, 2$ let δ_i be the map

$$\Lambda_i \oplus L \rightarrow V: (\lambda, \ell) \mapsto \lambda - \ell$$

and put $\chi_i := \dim_k(\ker \delta_i) - \dim_k(\operatorname{coker} \delta_i)$. Then

$$v(\det(\rho)) = \chi_1 - \chi_2$$

where $\det(\rho)$ is defined with respect to the $k[t]$ -structures given by Λ_1 and Λ_2 .

Remark 5 Equivalently one can define χ_i to be the Euler characteristic of the vector bundle on \mathbf{P}^1 defined by the triple (V, Λ_i, L) .

Proof of Lemma 2 Clearly $\chi_1 - \chi_2$ does not depend on the choice of L . Without loss of generality we may assume that $V = k((t^{-1}))^n$, $\Lambda_2 = k[t]^n$ and $L = k[[t^{-1}]]^n$ (so that (V, Λ_2, L) defines the vector bundle \mathcal{O}^n).

Since both $v(\det(\rho))$ and $\chi_1 - \chi_2$ are additive with respect to short exact sequences it suffices to verify their equality for $n = 1$ (since $K_0(\mathbf{P}^1)$ is generated by line bundles). For $n = 1$ it is clear that $v(\det(\rho)) = \chi_1 - \chi_2$.

Proof of Theorem 2 We need to show that

$$v(\operatorname{Reg}_R) = \dim_k H_R.$$

We will do so by computing $v(\operatorname{Reg}_R)$ using Lemma 2, applied to the $k[t]$ -modules $\exp^{-1}(E(R))$ and $\operatorname{Lie}_E(R)$ inside $V = \operatorname{Lie}_E(K_\infty)$. Choose $L \subset V$ small enough so that it satisfies Lemma 1.

Note that under the identification $\operatorname{Lie}_E(K_\infty) = K_\infty = E(K_\infty)$ the complex of k -vector spaces

$$\operatorname{Lie}_E(R) \oplus L \rightarrow \operatorname{Lie}_E(K_\infty)$$

coincides with

$$\delta_1: E(R) \oplus L' \rightarrow E(K_\infty).$$

To compute the valuation of the regulator we need to compare the Euler characteristic of δ_1 with that of

$$\delta_2: \exp^{-1}(E(R)) \oplus L \rightarrow \operatorname{Lie}_E(K_\infty).$$

We claim that there is an exact sequence

$$\ker \delta_1 \hookrightarrow \exp^{-1}(E(R)) \oplus L \xrightarrow{\delta_2} \operatorname{Lie}_E(K_\infty) \rightarrow \operatorname{coker} \delta_1 \twoheadrightarrow H_R \quad (3)$$

Together with Lemma 2 this directly implies Theorem 2. To construct this five-term sequence, consider the short exact sequence

$$E(R) \oplus L' \hookrightarrow \operatorname{Lie}_E(K_\infty) \oplus E(R) \oplus L' \twoheadrightarrow \operatorname{Lie}_E(K_\infty)$$

mapping to the short exact sequence

$$E(K_\infty) \hookrightarrow E(K_\infty) \twoheadrightarrow 0.$$

The resulting snake is the desired sequence (3).

The conjecture lends itself to numerical verification, and we have indeed verified for a dozen R of degree up to 31 over $\mathbf{F}_2[t]$. Below is the example of highest degree.

Example 1 Take $q = 2$, let K be the field of definition of the $(t^5 + t^2 + 1)$ -torsion of the Carlitz module, and R the integral closure of $k[t]$ in K . We computed

$$\frac{\zeta_R(1)}{\text{Reg}_R} = t^{20} + t^{17} + t^{15} + t^{14} + t^{13} + t^{11} + t^{10} + t^6 + t^4 + t + 1 + O(t^{-15}).$$

Note that H_R is isomorphic with

$$\frac{E(K_\infty)}{\exp_E(\text{Lie}_E(K_\infty)) + E(R)}.$$

This quotient can be computed by first taking the quotient of $E(K_\infty)$ by $L' + E(R)$ for some $L' \subset \exp_E(E(K_\infty))$ as in Lemma 1, which already gives something finite, and then modding out by successively more and more images of the exponential until the dimension agrees with the one predicted by Theorem 2. We computed the action of t on the quotient and found $|H_R| = t^{20} + t^{17} + \cdots + t + 1$, as predicted by the conjecture.

We end with a couple of remarks on Conjecture 1.

- Remark 6* 1. Conjecture 1 refines a conjecture given in [10]. The latter treats not only the Carlitz module, but also Drinfeld modules of higher rank that have everywhere good reduction, and it could be refined in a similar way.
2. If we interpret the complex $E(R) \rightarrow H^1(G, \Lambda_E)$ as an analogue of the complex $O_F^\times \rightarrow \{\pm 1\}^{r_1}$ then it appears as if there is a “class number” factor missing in Conjecture 1. However the module H_R can be interpreted as an Ext^2 of a “Carlitz shtuka” by a “trivial shtuka”, which in turn suggests that H_R is the “class module” (compare for example with $\text{Ext}_X^2(\mathbf{Z}, \mathbf{Z}(1)) = \text{Pic}(X)$ in motivic cohomology, see e.g. [7, p. 25]).
 3. Moreover, $\exp^{-1}(E(R))$ can be interpreted as the Ext^1 between the same objects. This suggests that it should be possible to interpret the v -adic (with $v \neq \infty$) special value formulas of Lafforgue [6] in a way similar to Conjecture 1. It also suggests that the techniques of [6] can be used to attack Conjecture 1.

We intend to return to these points in future papers.

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